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A CHARACTERIZATION OF THE RECTANGULAR DISTRIBUTION

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It is assumed that the statistics S and T (given by formula (1.2) and (1.3)) have constant regression on $\Lambda = \sum_{j=1}^{n} X_j$. Under certain additional assumptions this leads either to a symmetric two point distribution or to the rectangular distribution over $(-1, +1)$. The two point distribution can be excluded by assuming that the population distribution function is continuous. A characterization theorem for the rectangular distribution over $(-1, +1)$, is then obtained.

Rectangular (uniform) distribution; characterization; characteristic function; constant (zero) regression; symmetric two point distribution

1. Introduction and statement of the result

In this paper we derive a characterization of the rectangular (uniform) distribution over the interval $(-1, +1)$, that is, of the distribution

$$F(x) = \begin{cases} 0, & \text{if } \quad x < -1, \\ \frac{1}{2}(x+1), & \text{if } -1 \leq x < 1, \\ 1, & \text{if } \quad x \geq 1. \end{cases} \tag{1.1}$$

$F(x)$ has characteristic function $f(t) = (\sin t)/t$.

The characterization which we discuss here is based on a regression property which we define next.

Let X and Y be two random variables and assume that the conditional expectation $\mathscr{E}(Y|X)$ exists. We say that Y has constant regression on X if $\mathscr{E}(Y|X) = \mathscr{E}(Y)$ almost everywhere.

If $\mathscr{E}(Y) = 0$ and $\mathscr{E}(Y|X) = 0$, then we say that Y has zero regression on X.

The subsequent discussion utilizes the following lemma.

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Lemma 1.1. *Let X and Y be two random variables and assume that the expectation $\mathscr{E}(Y)$ exists. The random variable Y has constant regression on X if, and only if, the relation*

$$\mathscr{E}(Y\,\mathrm{e}^{\mathrm{i}tX}) = \mathscr{E}(Y)\mathscr{E}(\mathrm{e}^{\mathrm{i}tX})$$

holds for all real t.

For the proof we refer to [4, Chapter 6].

Our main result is the following theorem. In formulating it we write for brevity $\sum_{j\neq k}$ for $\sum_{j=1}^{n}\sum_{k=1,j\neq k}^{n}$ and $\sum_j$ instead of $\sum_{j=1}^{n}$.

Theorem. *Let $X_1, X_2, \ldots, X_n$ be a sample from a population with population distribution $F(x)$ and characteristic function $f(t)$. Suppose that*

(i) *$F(x)$ is non-degenerate,*

(ii) *$F(x)$ is symmetric,*

(iii) *the moment of order 4 of $F(x)$ exists.*

Let

$$S = \frac{1}{n(n-1)}\sum_{j\neq k}\{2X_j^3X_k - 4X_jX_k - 6X_j^2X_k^2\} + \frac{4}{n}\sum_j X_j^2, \tag{1.2}$$

$$T = \frac{1}{n(n-1)}\sum_{j\neq k}\{X_j^4X_k - 2X_j^2X_k^3\} + \frac{1}{n}\sum_j\{2x_j^3 - X_j\}, \tag{1.3}$$

$$\Lambda = \sum_j X_j \tag{1.4}$$

be three statistics and suppose that S as well as T has constant regression on Λ. Then $F(x)$ is

(a) *either a two point distribution whose only discontinuity points are the points $+1$ and -1 (on account of (ii) the jumps are both equal to $\frac{1}{2}$),*

(b) *or a uniform distribution over the interval $(-1, +1)$.*

Corollary 1. *If the assumptions of the theorem are satisfied and if in addition, one requires that*

(iv) *$F(x)$ is continuous,*

then $F(x)$ is the distribution (1.1) if, and only if, S as well as T have constant regression on Λ.

Corollary 2. *If the assumptions of the theorem are satisfied and if in addition one replaces assumption (iv) by*

(v) *$F(x)$ is purely discrete,*

then $F(x)$ is a symmetric two point distribution if, and only if, S as well as T has constant regression on Λ.

Corollary 1 gives the desired characterization of the uniform distribution over $(-1, +1)$.

Remark. This characterization is not the only characterization of the uniform distribution; for other characterizations see [1, 3] and [2, pp. 410 and 443]. However, these characterizations are of a different nature and use other properties (such as statements about order statistics, about maximum entropy, etc.). These authors employ also entirely different techniques.

2. The differential equations

According to the assumptions of the theorem S as well as T has constant regression on Λ. It follows therefore from Lemma 1.1 that

$$\mathscr{E}(S\,\mathrm{e}^{\mathrm{i}t\Lambda}) = \mathscr{E}(S)\mathscr{E}(\mathrm{e}^{\mathrm{i}t\Lambda}), \tag{2.1}$$

$$\mathscr{E}(T\,\mathrm{e}^{\mathrm{i}t\Lambda}) = \mathscr{E}(T)\mathscr{E}(\mathrm{e}^{\mathrm{i}t\Lambda}). \tag{2.2}$$

We see from assumption (ii) that $F(x)$ is symmetric, hence the moments of odd order of $F(x)$ vanish so that

$$\mathscr{E}(S) = 4\alpha_2 - 3\alpha_2^2 = \mathscr{C}(\text{say}), \quad \text{where } \alpha_2 = \int_{-\infty}^{\infty} x^2\,\mathrm{d}F(x), \tag{2.3}$$

$$\mathscr{E}(T) = 0. \tag{2.4}$$

Our regression assumptions have therefore the form

$$\mathscr{E}(S\,\mathrm{e}^{\mathrm{i}t\Lambda}) = \mathscr{C}[f(t)]^n, \tag{2.5a}$$

$$\mathscr{E}(T\,\mathrm{e}^{\mathrm{i}t\Lambda}) = 0 \tag{2.5b}$$

so that T has even zero regression on Λ. Our next aim is the transformation of the regression assumption (2.5a) and (2.5b) into differential equations for the characteristic function $f(t)$,

From the definition of $f(t)$ we see that

$$\mathscr{E}\Big[\sum_{j\neq k} X_j^3 X_k\,\mathrm{e}^{\mathrm{i}t\Lambda}\Big] = n(n-1)f'''f'(f)^{n-2}, \tag{2.6a}$$

$$\mathscr{E}\Big[\sum_{j\neq k} X_j X_k\,\mathrm{e}^{\mathrm{i}t\Lambda}\Big] = -\frac{n(n-1)}{2}(f')^2(f)^{n-2}, \tag{2.6b}$$

$$\mathscr{E}\Big[\sum_{j\neq k} X_j^2 X_k^2\,\mathrm{e}^{\mathrm{i}t\Lambda}\Big] = \frac{n(n-1)}{2}(f'')^2(f)^{n-2}, \tag{2.6c}$$

$$\mathscr{E}\Big[\sum_{j} X_j^2\,\mathrm{e}^{\mathrm{i}t\Lambda}\Big] = -nf''(f)^{n-1}. \tag{2.6d}$$

Similarily we obtain

$$\mathscr{E}\Big[\sum_{j\neq k} X_j^4 X_k \, e^{it\Lambda}\Big] = \frac{n(n-1)}{i} f^{(iv)} f'(f)^{n-2}, \tag{2.7a}$$

$$\mathscr{E}\Big[\sum_{j\neq k} X_j^2 X_k^3 \, e^{it\Lambda}\Big] = \frac{n(n-1)}{i} f'' f'''(f)^{n-2}, \tag{2.7b}$$

$$\mathscr{E}\Big[\sum_{j} X_j^3 \, e^{it\Lambda}\Big] = -\frac{n}{i} f'''(f)^{n-1}, \tag{2.7c}$$

$$\mathscr{E}\Big[\sum_{j} X_j \, e^{it\Lambda}\Big] = \frac{n}{i} f'(f)^{n-1}. \tag{2.7d}$$

It follows from (1.2) and from (2.6a)–(2.6d) that

$$\mathscr{E}(S\, e^{it\Lambda}) = 2f'''f'(f)^{n-2} + 2(f')^2(f)^{n-2} - 3(f'')^2(f)^{n-2} - 4f''(f)^{n-1}.$$

In a similar way we obtain from (1 3) and (2.7a)–(2.7d) the relation

$$\mathscr{E}(T\, e^{it\Lambda}) = \frac{1}{i} f^{(iv)} f'(f)^{n-2} - \frac{2}{i} f'' f'''(f)^{n-2} - \frac{2}{i} f'''(f)^{n-1} - \frac{1}{i} f'(f)^{n-1}.$$

Using (2.5a), (2.5b) and the last two equations one obtains the differential equations

$$2f'''f'(f)^{n-2} + 2(f')^2(f)^{n-2} - 3(f'')^2(f)^{n-2} - 4f''(f)^{n-1} = \mathscr{C}(f)^n,$$
$$f^{(iv)}f'(f)^{n-2} - 2f''f'''(f)^{n-2} - 2f'''(f)^{n-1} - f'(f)^{n-1} = 0.$$

Since f is a characteristic function there exists a neighborhood $\mathfrak{N}$ of the origin in which $f(t) \neq 0$. We restrict from now on t to this neighborhood and can divide the preceding two equations by $(f)^{n-2}$ and obtain

$$2f'''f' + 2(f')^2 - 3(f'')^2 - 4f''f = \mathscr{C}(f)^2, \tag{2.8}$$

$$f^{(iv)}f' - 2f''f''' - 2f'''f - f'f = 0. \tag{2.9}$$

3. Discussion and solution of the differential equations

One can rewrite (2.9) as

$$2f^{(iv)}f' + 2f'''f'' + 4f''f' - 6f'''f'' - 4f'''f - 4f''f' = 2f'f.$$

After integrating this equation one obtains

$$2f'''f' + 2(f')^2 - 3(f'')^2 - 4f''f = f^2 + \mathscr{C}_1. \tag{3.1}$$

From this and (2.8) one gets

$$\mathscr{C}(f)^2 = (f)^2 + \mathscr{C}_1$$

or

$$(\mathscr{C} - 1)(f)^2 = \mathscr{C}_1 \quad \text{for } t \in \mathfrak{N}.$$

We put $t=0$ and see that

$$\mathscr{C}-1=\mathscr{C}_1,$$

hence,

$$\mathscr{C}_1(f)^2=\mathscr{C}_1 \quad \text{or} \quad \mathscr{C}_1[(f)^2-1]=0.$$

Since, by assumption, $F(x)$ is nondegenerate so that $f=f(t)\not\equiv 1$, one has

$$\mathscr{C}_1=0, \qquad \mathscr{C}=1. \tag{3.2}$$

From (2.3) and (3.2) it follows that $3\alpha_2^2-4\alpha_2+1=0$ so that α_2 can assume only the values 1 or $\frac{1}{3}$. We see from (3.1) and (3.2) that we have to solve

$$2f'''f'+2(f')^2-3(f'')^2-4f''f=f^2. \tag{3.3}$$

It follows from (3.3) that

$$2f'(f'''+f')-(f''+f)^2-2f''(f''+f)=0. \tag{3.4}$$

Since

$$(f')^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f''+f}{f'}\right]=(f'''+f')f'-f''(f''+f),$$

we conclude from (3.4) that

$$2(f')^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f''+f'}{f'}\right]=(f''+f)^2. \tag{3.5}$$

We have to distinguish three cases:

Case I.

$$f''(0)+f(0)\not\equiv 0. \tag{3.6a}$$

In this case there exists a neighborhood $\mathfrak{N}_1$ of the origin in which neither $f(t)$ nor $f''(t)+f(t)$ vanish. We consider from now on $t\in\mathfrak{N}_1$. We divide (3.5) by $(f''+f)^2$ and get

$$2\left(\frac{f'}{f''+f}\right)^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f''+f}{f'}\right)=1. \tag{3.7}$$

We introduce $h(t)=(f''+f)/f'$; (3.7) then becomes $h^{-2}h'=\frac{1}{2}$ or $[h(t)]^{-1}=\frac{1}{2}t+c$, that is $f'(t)/(f''(t)+f(t))=-\frac{1}{2}t+c$. We put $t=0$ and see that $c=0$, hence $f'(t)/(f''(t)+f(t))=-\frac{1}{2}t$ or

$$tf''(t)+2f'(t)+tf(t)=0. \tag{3.8}$$

We write $g(t)=tf(t)$ and obtain from (3.8) the differential equation $g''(t)+g(t)=0$ which $g(t)$ must satisfy. Therefore

$$g(t)=\lambda_1\,\mathrm{e}^{\mathrm{i}t}+\lambda_2\,\mathrm{e}^{-\mathrm{i}t} \quad \text{and} \quad f(t)=\frac{\lambda_1\,\mathrm{e}^{\mathrm{i}t}+\lambda_2\,\mathrm{e}^{-\mathrm{i}t}}{t}.$$

From assumption (ii) we see that

$$f(t)=f(-t), \tag{3.9a}$$

moreover

$$f(0)=1. \tag{3.9b}$$

It follows from these two equations that

$$\lambda_1=\frac{1}{2i}, \qquad \lambda_2=\frac{1}{2i}$$

so that

$$f(t)=\frac{e^{it}-e^{-it}}{2it}=\frac{\sin t}{t}. \tag{3.10}$$

This formula was derived for $t\in\Re_1$ and can be extended (by using properties of analytic characteristic functions) to all values of t. This completes the proof of statement (b) of the theorem.

Case II.

$$f''(0)+f(0)=0, \tag{3.6b}$$

so that

$$\alpha_2=1. \tag{3.11}$$

Case II(a). There exists an a such that $f''(t)+f(t)\neq 0$ for $t\in[-a,a]\backslash\{0\}$, i.e. in an interval punctured at the origin. We consider the half open interval $(0,a]$ and apply the reasoning used in studying Case I and see that $f(t)=(1/t)\sin t$ for $0<t\leq a$. This is also true in the interval $-a\leq t<0$. Since $f(0)=1$ and $\lim_{t\to 0}(1/t)\sin t=1$ we conclude that $f(t)=(1/t)\sin t$ for $|t|\leq a$. But then $\alpha_2=\frac{1}{3}$ in contradiction to (3.11) so that Case II(a) is excluded.

Case II(b). There exists a sequence $\{t_v\}$ such that $t_v\to 0$ and $f''(t_v)+f(t_v)=0$ while $f''(t)+f(t)\neq 0$ for $t_{v+1}<t<t_v$, $v=1,2,\ldots$. We select a v such that $t_v<\pi$. Using again the argument employed in Case I we see that $f(t)=(1/t)\sin t$ for $t_{v+1}<t<t_v$. Then

$$f''(t)+f(t)=2\left(\frac{\sin t-t\cos t}{t^3}\right) \quad \text{for } t_{v+1}<t<t_v.$$

However, this expression becomes $\frac{2}{3}$ at $t=0$. This is in contradiction to (3.6b) so that Case II(b) is also excluded.

Case III. Assume that (3.6b) holds, but that there exists an interval $|t|<a$, where

$$f''(t)+f(t)=0. \tag{3.6c}$$

Then (3.5) is satisfied and

$$f(t) = \lambda_1 e^{it} + \lambda_2 e^{-it}.$$

It follows then from (3.9a) and (3.9b) that $\lambda_1 = \lambda_2 = \frac{1}{2}$, hence,

$$f(t) = \cos t, \quad t \in (-a, a). \tag{3.12}$$

The validity of (3.12) can again be extended to all values of t. Eq. (3.12) is the characteristic function of a symmetric two point distribution so that statement (a) of the theorem is proved.

We still have to prove the corollaries. Assumption (iv) of Corollary 1 excludes the solution (3.12). The sufficiency of the conditions of the corollary follows from the theorem. To prove the necessity we assume that $f(t) = (1/t) \sin t$. This function satisfies the differential equations (2.8) and (2.9). Using (2.6a) to (2.6d) and Lemma 1.1 one obtains (2.1). Relation (2.2) follows from (2.7a) to (2.7d) and Lemma 1.1. Corollary 2 is proved in the same way.

References

[1] S.G. Ghurye, Characterization of some location and scale parameter families of distributions, in: Contributions to Probability and Statistics, Essays in honor of Harold Hotelling (Stanford University Press, Stanford, CA, 1960) 203–215.

[2] A.M. Kagan, Yu.V. Linnik and C.R. Rao, Characterization Problems in Mathematical Statistics (Wiley, New York, 1973).

[3] S. Kotz, Characterizations of statistical distributions, a supplement to recent surveys. Internat. Statist. Rev. 42 (1974) 39–65.

[4] E. Lukacs and R.G. Laha, Applications of Characteristic Functions (Griffin, London, 1964).